(4) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. We can differentiate power series term by term. We have

$$\cos z = \frac{d}{dz}\sin z = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

(5) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$. Recall $\sinh z = -i \sin iz$. Hence,

$$\sinh z = -i \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} i^{2n+2}}{(2n+1)!} z^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (-1)^{n+1} z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

(6) $\cosh z = \frac{d}{dz}\sinh z = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$.

$\square$

Note: the power series in (2)-(6) are the usual Maclaurin series when z is real. This provides additional justification that we chose the correct definitions when extending the elementary functions to the complex plane.

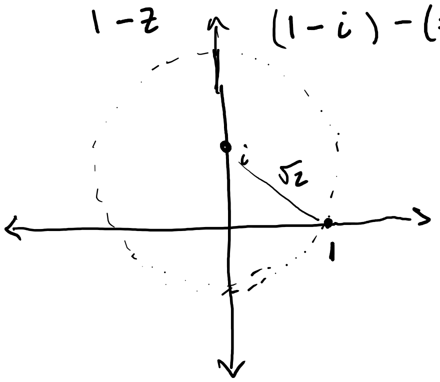
$\boxed{\text{Example}}$ We use the six MacLauren Series for the elementary functions to compute MacLauren Series or Taylor Series of other functions.

(a) MacLauren series of $\frac{1}{1+z}$. We have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} \overset{(1)}{=} \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \qquad \left(|z| < 1 \right)$$

(b) Taylor series for $\frac{1}{1-z}$ about $z_0 = i$. We have

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1 - \left(\frac{z-i}{1-i}\right)}$$



$$\overset{(1)}{=} \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n \qquad \left(|z-i| < |1-i| = \sqrt{2} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}.$$

(c) MacLauren series of $z^2 e^{2z}$. We have

$$z^2 e^{2z} \overset{(2)}{=} z^2 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^{n+2}}{n!}$$

$$= \sum_{n=2}^{\infty} \frac{2^{n-2} z^n}{(n-2)!}. \qquad //$$

Laurent Series

When f is not analytic at a point z_0, Taylors theorem cannot be applied. However, we can often find a series representation of f that involves negative powers of $z-z_0$.

Examples

(1) $f(z) = \dfrac{e^{-z}}{z^2}$. The function is not analytic at $z_0 = 0$ so we look for a series expansion involving powers of z. We have

$$\frac{e^{-z}}{z^2} \overset{(2)}{=} \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n-2}}{n!}$$

$$= \frac{1}{z^2} - \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+2)!}$$

$$(0 < |z| < \infty)$$

(2) $f(z) = \dfrac{1+2z^2}{z^3+z^5}$.

$$\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) = \frac{1}{z^3}\left(\frac{2(1+z^2)-1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(\frac{2(1+z^2)}{1+z^2} - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

$$\overset{(1)}{=} \frac{2}{z^3} - \frac{1}{z^3}\sum_{n=0}^{\infty}(-z^2)^n \qquad (0 < |z| < 1)$$

$$= \frac{2}{z^3} - \sum_{n=0}^{\infty}(-1)^n z^{2n-3}$$

$$= \frac{2}{z^3} - \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty}(-1)^n z^{2n-3}$$

$$= \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty} (-1)^n z^{2n-3}.$$

(3) $f(z) = \dfrac{e^z}{(z+1)^2}$. The singularity is at $z_0 = -1$ so we are looking for powers of $z+1$. We have

$$\frac{e^z}{(z+1)^2} = \frac{e^{z+1}}{e(z+1)^2} = \frac{1}{e(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!} \qquad \left(0 < |z+1| < \infty\right)$$

$$= \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!}\right] \quad //$$

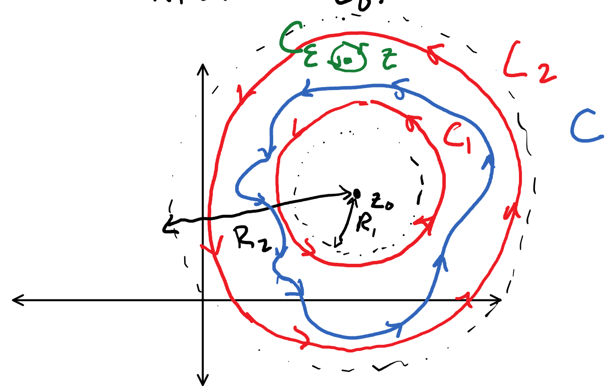
$\boxed{\text{Theorem (Laurent)}}$ Suppose that f is analytic on an annulus $R_1 < |z - z_0| < R_2$. Then f has a Laurent Series representation on that annulus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \qquad \left(R_1 < |z - z_0| < R_2\right)$$

with coefficients given by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz$$

where C is an positively oriented simple closed contour in the annulus whose interior contains z_0.



Proof. Initially, assume $z_0 = 0$. Let z be such that $R_1 < |z| < R_2$. Let C_1 and C_2 be circles (w/ positive orientation) with radii r_1 and r_2 such that

$$R_1 < r_1 < |z| < r_2 < R_2$$

and such that the contour C lies in between C_1 and C_2. Also let $\varepsilon > 0$ be so small that the circle $C_\varepsilon = C_\varepsilon(z)$ lies in between C_1 and C_2.

Now, we compute the remainder. First,

$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(s)}{s-z}\, ds \qquad \left(\begin{array}{c}\text{Cauchy Int.}\\ \text{Formula}\end{array}\right)$$

$$= \frac{1}{2\pi i} \left(\int_{C_2} \frac{f(s)}{s-z}\, ds + \int_{C_1} \frac{f(s)}{z-s}\, ds \right) \qquad \left(\begin{array}{c}\text{Cauchy}\\ \text{Goursat}\end{array}\right).$$

Recall, from proof of Taylor's Theorem:

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + z^N \frac{1}{(s-z)s^N}$$

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + s^N \frac{1}{(z-s)z^N}$$

$$= \sum_{n=1}^{N} \frac{s^{n-1}}{z^n} + s^N \frac{1}{(z-s)z^N}.$$

Then

$$\rho_N(z) = f(z) - \sum_{n=0}^{N-1} a_n (z-0)^n + \sum_{n=1}^{N} b_n \frac{1}{(z-0)^n}$$

$$= \frac{1}{2\pi i} \left(\int_{C_2} f(s) \left(\frac{1}{s-z} - \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} \right) ds \right.$$

$$\left. + \int_{C_1} f(s) \left(\frac{1}{z-s} - \sum_{n=1}^{N} \frac{z^{-n}}{s^{-n+1}} \right) ds \right)$$

$$= \frac{1}{2\pi i}\left(\int_{C_2} \frac{f(s)\, z^N}{(s-z)\, s^N} + \int_C \frac{f(s)\, s^N}{(z-s)\, z^N} \right)$$

Then
$$|\rho_N(z)| \le \frac{1}{2\pi}\left| \int_{C_2} \frac{f(s) z^N}{(s-z) s^N}\, ds \right| + \frac{1}{2\pi}\left| \int_C \frac{f(s) s^N}{(z-s) z^N}\, ds \right|$$

You can show that both integrals on the right converge to 0 as $N \to \infty$ using the T.I. for Contour integrals, as in the proof of Taylors theorem. This proves the claim when $z_0 = 0$.

Suppose $z_0 \neq 0$ and assume f satisfies the conditions of the theorem. Define $g(z) = f(z+z_0)$. Since f is analytic on $R_1 < |z-z_0| < R_2$, g is analytic on $R_1 < |z| < R_2$. By the case we just proved,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$$

with
$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z^{n+1}}\, dz \qquad b_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z^{-n+1}}\, dz$$

where Γ is the contour obtained from C by translation by z_0. To finish the proof, replace g by $f(z+z_0)$ and replace z by $z - z_0$. This completes the proof. ▨

$\boxed{\text{Example}}$ Laurent series are rarely found by using the integral expressions. Usually they are found by making use of the 6 MacLauren series for elementary functions.

(1) $f(z) = \dfrac{1}{z(1+z^2)}$. The singularities are at $0, i, -i$,

so the function is analytic on $0 < |z| < 1$. By Laurent's theorem, f has a Laurent series on this annulus. We have

$$\frac{1}{z(1+z^2)} = \frac{1}{z}\left(\frac{1}{1+z^2}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n-1}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1}.$$

Fun fact: $b_1 = 1$ so by the Laurent theorem

$$2\pi i = 2\pi i\, b_1 = \int_C f(z)\,dz = \int_C \frac{1}{z(1+z^2)}\,dz$$

where C is any positively oriented simple closed contour about 0 in the annulus.

(2) $f(z) = e^{1/z}$. The singularity is at $z_0 = 0$. The function is analytic on $0 < |z| < \infty$. We have

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{z^n n!} = 1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \cdots.$$

Notice that $b_1 = 1$ so by Laurent's theorem

$$2\pi i = 2\pi i\, b_1 = \int_C e^{1/z}\,dz \quad \text{where}$$

C is any simple closed contour about 0.